SECTION 15.5: DIRECTIONAL DERIVATIVES AND THE GRADIENT

RECALL: Given a function of two variables $f(x, y)$:

- the **partial** derivative of f **with respect to** x is $\dfrac{\partial f}{\partial x} = f_x(x, y) = \lim\limits_{h \to 0} \dfrac{f(x + h, y) - f(x, y)}{h}$
- the **partial** derivative of f **with respect to** y is $\dfrac{\partial f}{\partial y} = f_y(x, y) = \lim\limits_{k \to 0} \dfrac{f(x, y + k) - f(x, y)}{k}$

Loosely speaking, $f_x(a, b)$ is the 'derivative in the x-direction' at $(a, b, f(a, b))$ while $f_y(a, b)$ is the 'derivative in the y-direction' at $(a, b, f(a, b))$.

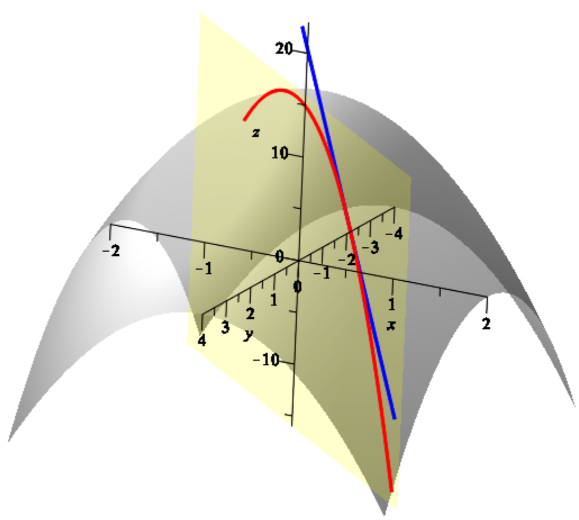
DEFINITION: Given a function $f(x, y)$ and an angle θ, let $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$.

The **derivative of f in the direction of $\hat{u}$** is:

$$D_{\hat{u}} f(a, b) = \lim_{t \to 0} \frac{f(a + t\cos(\theta), b + t\sin(\theta)) - f(a, b)}{t},$$

provided this limit exists.

RECALL: The (parametric) description of a line containing (a, b) with direction $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$ is precisely $x = a + t\cos(\theta)$ and $y = b + t\sin(\theta)$. So we are literally 'slicing' the surface $z = f(x, y)$ with the plane determined by the line $x = a + t\cos(\theta)$ and $y = b + t\sin(\theta)$ in the xy-plane.



EXAMPLE 1: What happens if $\theta = 0$ in the definition above? What happens if $\theta = \frac{\pi}{2}$?

Ans: $\theta = 0$ results in $f_x(x, y)$ and $\theta = \dfrac{\pi}{2}$ results in $f_y(x, y)$.

THEOREM: If f is differentiable and $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$, then $D_{\hat{u}} f(a, b) = f_x(a, b) \cos(\theta) + f_y(a, b) \sin(\theta)$.

PROOF: Use the chain rule with $x = a + t\cos(\theta)$ and $y = b + t\sin(\theta)$.

EXAMPLE 2: Let $f(x, y) = e^{2x+y} \cos(y)$ and $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$.

1. Prove f is differentiable at $(0, 0)$ by analyzing $f_x(x, y)$ and $f_y(x, y)$.

   Ans: $f_x(x, y) = 2e^{2x+y} \cos(y)$ and $f_y(x, y) = e^{2x+y} \cos(y) - e^{2x+y} \sin(y)$ are both continuous everywhere.

2. Find an expression for $D_{\hat{u}} f(0, 0)$ in terms of θ.

   Ans: $D_{\hat{u}} f(0, 0) = 2\cos(\theta) + \sin(\theta)$

3. Find $D_{\hat{u}} f(0, 0)$ for $\theta = \frac{\pi}{6}$, $\theta = \frac{3\pi}{4}$, and $\theta = \frac{3\pi}{2}$.

   Ans: $\theta = \frac{\pi}{6}$, $D_{\hat{u}} f(0, 0) = \sqrt{3} + \frac{1}{2}$; $\theta = \frac{3\pi}{4}$, $D_{\hat{u}} f(0, 0) = -\frac{\sqrt{2}}{2}$; $\theta = \frac{3\pi}{2}$, $D_{\hat{u}} f(0, 0) = -1$

4. Use a graphing utility to approximate the values of θ which maximize and minimize $D_{\hat{u}} f(0, 0)$. What are the maximum and minimum values of $D_{\hat{u}} f(0, 0)$? (For a challenge, rewrite $D_{\hat{u}} f(0, 0)$ as a sinusoid of the form $A \sin(\omega\theta + \phi)$ and find the maximum and minimum values that way ...)

   Ans: max is approx. 2.236 when $\theta \approx 0.464$; min is approx. -2.236 when $\theta \approx 3.605$

Since $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$, the formula $D_{\hat{u}} f(a, b) = f_x(a, b) \cos(\theta) + f_y(a, b) \sin(\theta)$ can be viewed as a dot product:

$$D_{\hat{u}} f(a, b) = f_x(a, b) \cos(\theta) + f_y(a, b) \sin(\theta) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \hat{u}$$

The vector $\langle f_x(a, b), f_y(a, b) \rangle$ is given a special name:

DEFINITION: If $f_x(a, b)$ and $f_y(a, b)$ exist, the **gradient of** f is the vector $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$.

DIRECTIONAL DERIVATIVE REPRISE If f is differentiable at (a, b) and $\hat{u} = \langle \cos(\theta), \sin(\theta) \rangle$, then

$$D_{\hat{u}} f(a, b) = \nabla f(a, b) \cdot \hat{u}$$

That is, $D_{\hat{u}} f(a, b)$ is how much of the gradient is in the direction of $\hat{u}$ at (a, b).

PROPERTIES OF THE GRADIENT: If f is a differentiable at (a, b) then:

- If $\nabla f(a, b) \neq \vec{0}$, then $\nabla f(a, b)$ points in the direction of greatest increase.

  That is, $D_{\hat{u}} f(a, b)$ is largest when $\hat{u} = \widehat{\nabla f(a, b)}$, and the largest value of $D_{\hat{u}} f(a, b)$ is $\|\nabla f(a, b)\|$.
- If $\nabla f(a, b) \neq \vec{0}$, then $-\nabla f(a, b)$ points in the direction of greatest decrease.

  That is, $D_{\hat{u}} f(a, b)$ is smallest when $\hat{u} = -\widehat{\nabla f(a, b)}$, and the smallest value of $D_{\hat{u}} f(a, b)$ is $-\|\nabla f(a, b)\|$.
- If $\nabla f(a, b) = \vec{0}$, then $D_{\hat{u}} f(a, b) = 0$ for all directions $\hat{u}$.
- **ORTHOGONALITY:** $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = f(a, b)$ at the point (a, b).

PROOF:

EXAMPLE 3: Let $f(x,y) = e^{2x+y}\cos(y)$.

1. Find an expression of $\nabla f(x,y)$.

Ans: $\nabla f(x,y) = \left\langle 2e^{2x+y}\cos(y), e^{2x+y}\cos(y) - e^{2x+y}\sin(y)\right\rangle$

2. Find $\nabla f(0,0)$ and use $\nabla f(0,0)$ to find the maximum and minimum values of $D_{\hat{u}} f(0,0)$.

Ans: $\nabla f(0,0) = \langle 2,1\rangle$ so max. is $\|\langle 2,1\rangle\| = \sqrt{5}$ and min. is $-\|\langle 2,1\rangle\| = -\sqrt{5}$

3. Find the angles θ which correspond to $\widehat{\nabla f(0,0)}$ and $-\widehat{\nabla f(0,0)}$.

Ans: $\widehat{\nabla f(0,0)} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$ so $\theta = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$; $-\widehat{\nabla f(0,0)} = \left\langle -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$ so $\theta = 2\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$

4. Compare your answers to the two previous parts to what you found earlier when you optimized $D_{\hat{u}} f(0,0)$.

Ans: $\pm\sqrt{5} \approx \pm 2.236$, $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 0.464$, and $2\pi - \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 3.605$ ✓

5. Find the directions $\hat{u}$ at $(0,0)$ in which there is no change in f.

Ans: Any direction orthogonal to $\widehat{\nabla f(0,0)} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right\rangle$: $\hat{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, \right\rangle$ or $\hat{u} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$

EXAMPLE 4: Let $f(x,y) = \dfrac{8y}{1+x^2+y^2}$.

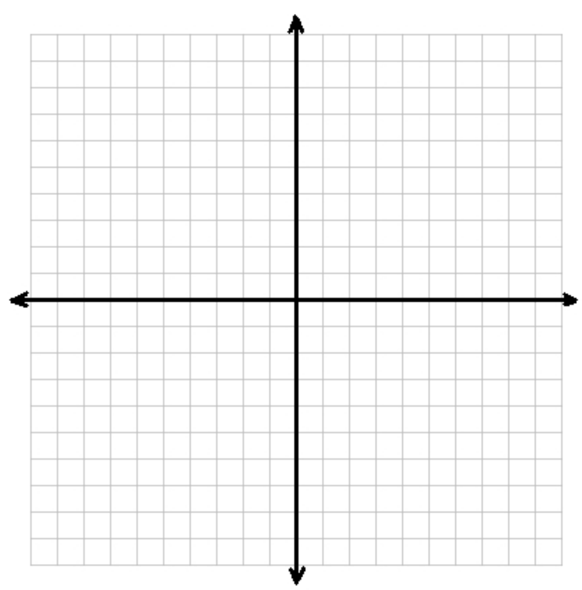
1. Find and simplify $f_x(x,y)$ and $f_y(x,y)$ and use these to explain why f is differentiable at $(\sqrt{3}, 2)$.

Ans: $f_x(x,y) = -\dfrac{16xy}{(1+x^2+y^2)^2}$ and $f_y(x,y) = \dfrac{8x^2 - 8y^2 + 8}{(1+x^2+y^2)^2}$

2. Find and simplify $\nabla f(\sqrt{3}, 2)$.

Ans: $\nabla f(\sqrt{3}, 2) = \left\langle -\dfrac{\sqrt{3}}{2}, 0 \right\rangle$

3. Graph the level curve corresponding to $f(x,y) = f(\sqrt{3}, 2)$. Plot $\nabla f(\sqrt{3}, 2)$ with initial point $(\sqrt{3}, 2)$.



EXAMPLE 5: Recall the function Q below is the amount of units produced annually, in *thousands*, where x and y represent the amount of money, in *millions of dollars*, spent annually on labor and capital, respectively. Currently, \$1 million is spent on labor and \$8 million is spent on capital.

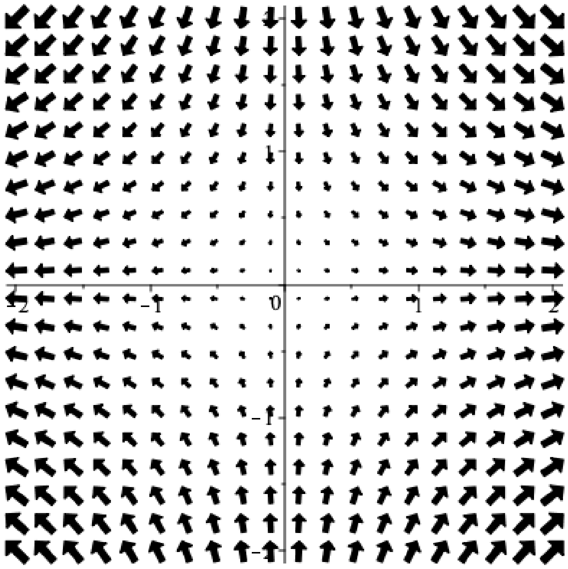
$$Q(x,y) = 150x^{1/3}y^{2/3}, \quad x > 0, \quad y > 0$$

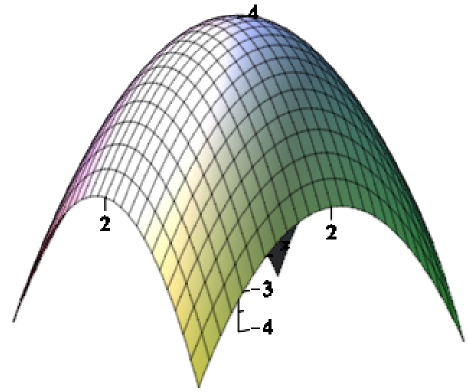
It turns out that \$40 million is available for investment in the company. The natural question is, of course, how to best invest this money to maximize the resulting production. Recall $\nabla Q(1,8)$ points in the direction of the maximum increase of Q. Use $\nabla Q(1,8)$ to estimate how to portion out the \$40 million to effect the greatest increase in production. Find the resulting production.

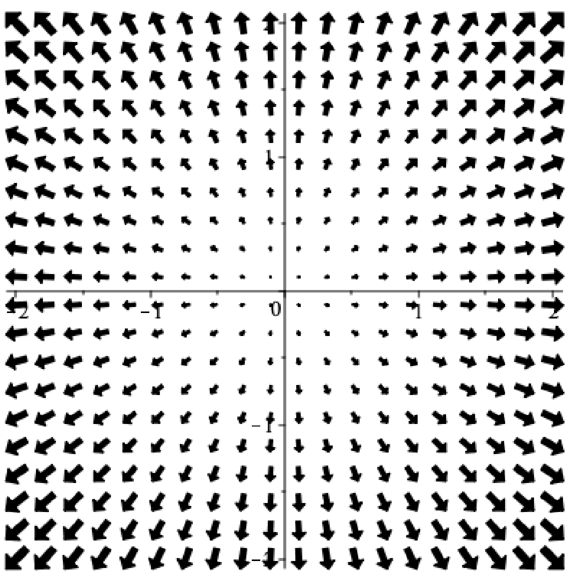
Ans: $\nabla Q(1,8) = \langle 200, 50\rangle$ so $\dfrac{\Delta y}{\Delta x} = \dfrac{50}{200} = \dfrac{1}{4}$ so $\Delta x = 4\Delta y$. If $\Delta x + \Delta y = 40$, we get $\Delta x = 32$ and $\Delta y = 8$.

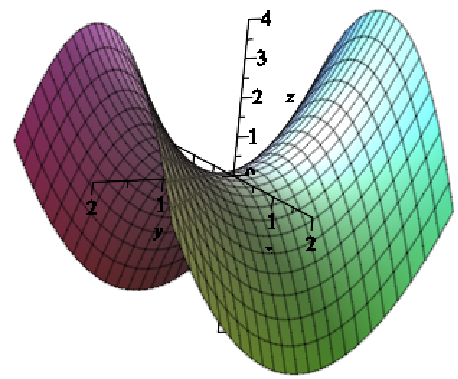
$Q(1+32, 8+8) = Q(33,16) \approx 3054.986$ which corresponds to 3,054,986 units.

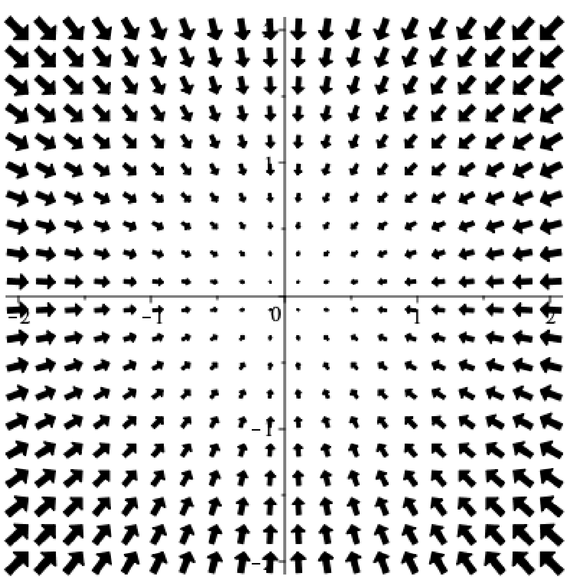
EXAMPLE 6: Match the gradient plot below on the left with the corresponding surface on the right.

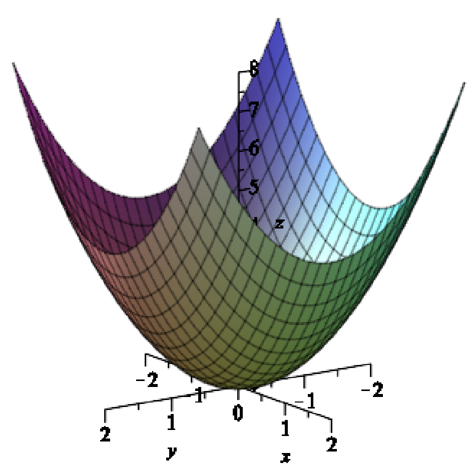












GRADIENTS OF FUNCTIONS OF THREE VARIABLES

EXAMPLE 7: Suppose F is a differentiable function of three variables. How would you define $\nabla F(a, b, c)$?

ORTHOGONALITY: $\nabla F(a, b, c)$ is normal to the **level surface** $F(x, y, z) = F(a, b, c)$ at (a, b, c).

EXAMPLE 8: Consider the surface defined as: $z\, e^{x+2y} = 1$.

1. Let $F(x, y, z) = z\, e^{x+2y}$. Prove F is differentiable at $\left(-1, \frac{1}{2}, 1\right)$ by analyzing the first partials of F.

   Ans: $F_x(x, y, z) = z\, e^{x+2y}$, $F_y(x, y, z) = 2z\, e^{x+2y}$, $F_z(x, y, z) = e^{x+2y}$. All are continuous everywhere.

2. Find $\nabla F\left(-1, \frac{1}{2}, 1\right)$ and use this to find the equation of the tangent plane to $z\, e^{x+2y} = 1$ at $\left(-1, \frac{1}{2}, 1\right)$.

   Ans: $\nabla F\left(-1, \frac{1}{2}, 1\right) = \langle 1, 2, 1\rangle$; Tangent plane: $x + 2y + z = 1$.

3. Check your answer graphically. **NOTE:** On geogebra, you'll need to solve $ze^{x+2y} = 1$ for z.

TANGENT PLANES REVISITED:

Suppose f is differentiable and consider the surface: $z = f(x, y)$.

We can rewrite this equation as: $z - f(x, y) = 0$.

If we let $F(x, y, z) = z - f(x, y)$, we can think of the graph of $z - f(x, y) = 0$ as the level surface $F(x, y, z) = 0$.

Calculating $\nabla F(x, y, z)$, we find $\nabla F(x, y, z) = \langle -f_x(x, y), -f_y(x, y), 1\rangle$.

Does this look familiar? This is the same normal vector we arrived at when we first studied tangent planes!

HOMEWORK: Section 15.5: 15 - 81 every other odd, 85 - 91 odd*; Section 15.6: 13, 15, 21, 23, 31